JOURNAL OF APPROXIMATION THEORY 44, 30–42 (1985)

Relative Differentiability and Integral Representation of a Class of Weak Markov Systems

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Communicated by E. W. Cheney

Received September 1, 1983

Integral representations for Markov systems have been given under various hypotheses. It is shown that two of these representations do not hold in general, and an improved version is given which is more general than existing results. The proof is based on a number of new results concerning weak Markov systems which may be of independent interest. © 1985 Academic Press, Inc.

0. INTRODUCTION

Let M be a subset of the real line $\mathbb{R}$ of cardinality $\geqq n+2$, $F(M)=\{f\colon M\to\mathbb{R}\}$, $f_0, f_1,..., f_n \in F(M)$ and $U_i=\text{span}\{f_0,..., f_i\}$ for $i=0, 1,..., n$. $f_0,..., f_n$ is called a (weak) Čebyšev system—and U_n is called a (weak) Haar space—iff for no $f\in U_n\backslash\{0\}$ there exist points $t_0,..., t_{n+1}\in M$ with $t_0<\cdots<t_{n+1}$ and $(-1)^i f(t_i)\geqq 0$ $((-1)^i f(t_i)>0)$ for $i=0, 1,..., n+1$. An equivalent formulation is that $\det(f_i(t_j))$ has strictly (weakly) constant sign for every choice $t_0,..., t_n\in M$ with $t_0<\cdots<t_n$ (see [10]).

$f_0,..., f_n$ is called a (weak) Markov system iff $f_0,..., f_i$ is a (weak) Čebyšev system for $i=0, 1,..., n$. If, in addition, $f_0\equiv 1$, we speak of a normed (weak) Markov system.

In 1965, Rutman [2] gave the following integral representation for normed Markov systems with only hints of the proof:

THEOREM 1. *If M is an open interval and U_n is spanned by a normed Markov system of right-continuous functions, there exists a basis $g_0,..., g_n$ of U_n and right-continuous strictly increasing functions $w_1,..., w_n\in F(M)$ and $c\in M$ such that for all $x\in M$ one has*

0021-9045/85 $3.00

$$g_0(x) \equiv 1$$

$$g_1(x) = \int_c^x dw_1(t_1)$$

$$g_2(x) = \int_c^x \int_c^{t_1} dw_2(t_2)\, dw_1(t_1)$$

$$\vdots$$

$$g_n(x) = \int_c^x \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1).$$

It was observed by the author (see [3]) and independently by Zalik [6] that there are counterexamples to Theorem 1. Zalik [6] gave a more complicated integral representation under weaker hypotheses. His proof is based on a well-known Gauss kernel approximation for Markov systems by smooth Markov systems for which Rutman's representation holds. However, there are counterexamples to this representation, too, as will be shown in the next section.

Our main result is a corrected version of this integral representation which at the same time is more general in that it requires weaker hypotheses on M and holds for a wide class of weak Markov systems. Our proof does not use the Gauss kernel approximation but is based on several new properties of weak Markov systems some of which are analogous to the relative differentiability of Markov systems derived in [9].

1. An Example

We start with a

Definition. M has property (B) if it has no smallest or largest element and if for each pair $x, y \in M$ with $x < y$ there is a $z \in M$ with $x < z < y$. In [6], essentially the following result is stated:

Theorem 2. *Let M be a nondenumerable set with property (B) and $c \in M$. Let U_n be spanned by a normed Markov system.*

(a) *Then there exist a basis $g_0, ..., g_n$ of U_n, a subset B of M such that $M \backslash B$ is denumerable, a strictly increasing function $h \in F(M)$ and strictly increasing functions $w_1, ..., w_n \in F((\inf h(M), \sup h(M))$ with $w_i(h(c)) = 0$, $i = 1, ..., n$, such that for all $x \in B$ one has*

$$g_0 \equiv 1$$

$$g_1(x) = \int_{h(c)}^{h(x)} dw_1(t_1)$$

$$\vdots$$

$$g_n(x) = \int_{h(c)}^{h(x)} \int_{h(c)}^{t_1} \cdots \int_{h(c)}^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1).$$

Moreover, the w_i*'s are uniquely determined a.e. by* $h, g_1, \ldots, g_n$.

(b) *If, in addition, M is a dense subset of an open interval, one may choose* $h(x) = x$ *for* $x \in M$.

Statement (b) implies that Rutman's result is correct for all but denumerably many points of M. We consider the following

EXAMPLE. Let a normed Markov system on $M = \mathbb{R}$ be defined by

$$\begin{aligned} f_1(t) &= t && \text{for } t < 1 \\ &= t + 1 && \text{for } t \geqq 1 \\ f_2(t) &= \frac{t^2}{2} && \text{for } t < 1 \\ &= \frac{(t+1)^2}{2} && \text{for } t \geqq 1. \end{aligned}$$

After multiplication with suitable constants we have

$$g_1 = \alpha + f_1 \quad \text{and} \quad g_2 = \beta + \gamma f_1 + f_2 \quad \text{for some } \alpha, \beta, \gamma \in \mathbb{R}.$$

According to Theorem 2, we choose $h(x) \equiv x$ and $c = 0$. The integral representation implies $\alpha = \beta = 0$, so $g_1 = f_1$,

$$f_2(x) = \int_0^x \left(-\gamma + \int_0^t dw_2(t_2) \right) dw_1(t_1),$$

and

$$\begin{aligned} w_1(t) &= t && \text{for } t < 1 \\ &= t + 1 && \text{for } t > 1 \end{aligned}, \qquad dw_1(1) = 1.$$

From $f_2'(x) = w_1(x)$ for $x \neq 1$ follows $\gamma = 0$, so $g_2 = f_2$ and $w_2(x) = w_1(x)$ for $x \neq 1$. As w_1 and w_2 are strictly increasing, we have $w_1(1)$, $w_2(1) \in \{1, 2\}$.

But this implies

$$f_2(x) = \int_0^x \int_0^t dw_2(t_2)\, dw_1(t_1) = \int_0^x w_2(t)\, dw_1(t)$$

$$= \frac{x^2}{2} \qquad \text{for} \quad x < 1$$

$$= \frac{(x+1)^2}{2} - \frac{1}{2} \qquad \text{for} \quad x \geqq 1, \text{ if } w_2(1) = 1$$

$$= \frac{(x+1)^2}{2} + \frac{1}{2} \qquad \text{for} \quad x \geqq 1, \text{ if } w_2(1) = 2.$$

So we get a contradiction for every $x \geqq 1$, i.e., for a nondenumerable set of points. The reason why Theorem 2 does not hold is that jump discontinuities like the one considered cannot be dealt with appropriately by repeated Riemann integrations.

2. Results

We shall call a weak Haar space $U \subset F(M)$ nondegenerate iff for every $a \in M$, the spaces $U|_{(-\infty,a)\cap M}$ and $U|_{(a,\infty)\cap M}$ have the same dimension as U.[1] A weak Markov system $f_0, f_1, ..., f_n \in F(M)$ is called nondegenerate iff U_n is nondegenerate (this implies that each U_i, $0 \leqq i \leqq n$, is nondegenerate). If $U \subset F(M)$ is a nondegenerate Haar space of positive dimension, one obviously has $\inf M \notin M$, $\sup M \notin M$. Our main result is

THEOREM 3. *Let $f_0, f_1, ..., f_n \in F(M)$ be a nondegenerate normed weak Markov system. Then there exists a basis $g_0, ..., g_n$ of U_n, a strictly increasing function $h \in F(M)$, continuous increasing functions $w_1, ..., w_n$ defined on $J :=$ $(\inf h(M), \sup h(M))$ and $c \in J$ such that for all $x \in M$ we have*

$$g_0 \equiv 1$$

$$g_1(x) = \int_c^{h(x)} dw_1(t_1) \qquad (*)$$

$$\vdots$$

$$g_n(x) = \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1).$$

[1] Our definition of degeneracy is slightly different from the definition used by Zwick [11].

COROLLARY 3. *Let the hypotheses of Theorem* 3 *be fulfilled,* $A \subset M$ *a set with property* (*B*) *and* $f_0, f_1, ..., f_n$ *a normed Markov system on* A. *Then the* $w_1, ..., w_n$ *from Theorem* 3 *may be chosen such that their restrictions to* A *are strictly increasing.*

Remarks. (1) Even if M is an open interval and $f_0, f_1, ..., f_n$ a Markov system we cannot assume h to be the identity function.

(2) It is easy to show that if $g_0, g_1, ..., g_n$ have a representation (*), where h and $w_1, ..., w_n$ have the properties stated in Corollary 3′ (Theorem 3), they form a normed (weak) Markov system on M (see [7]).

(3) If M does not contain inf M, sup M, any (weak) Haar space in $F(M)$ has a (weak) Markov basis (see [7, 5], also [10]).

(4) Any Markov system $f_0, ..., f_n$ may be transformed into a normed Markov system by division by f_0.

(5) Čebyšev systems defined on arbitrary totally ordered sets may be transformed to equivalent systems defined on subsets of $\mathbb{R}$. This transformation preserves property (B) (see [4]).

(6) Zalik has communicated a different proof of Theorem 3 based on a new embedding property of weak Haar spaces [8].

For the proof of Theorem 3, we need some notations and a number of auxiliary results. For $k \in \mathbb{N}$, let $\Delta_k(M) = \{(t_1, ..., t_k) \in M^k \mid t_1 < \cdots < t_k\}$.

DEFINITION. An $f \in F(M)$ has a strong oscillation of length k iff there exist $(t_1, ..., t_k) \in \Delta_k(M)$ and $h \in \{-f, f\}$ such that

$$h(t_1) < h(t_2) > h(t_3) < h(t_4) > \cdots.$$

The following is a generalization of a theorem of Zwick [12], which in turn is a generalization of Theorem 8.8 in [10]:

LEMMA 1. *Let* $f_0, ..., f_n \in F(M)$ *be linearly independent with* $f_0 = 1$. *Then the following properties are equivalent*:

(a) $f_0, ..., f_n$ *is a* (*normed*) *weak Markov system.*

(b) *No* $f \in U_n$ *has a strong oscillation of length* $n+2$.

Proof. (b) ⇒ (a) By Theorem 8.3 in [10] there exist normed i-dimensional spaces V_i, $i = 1, ..., n$, with $V_1 \subset V_2 \subset \cdots \subset V_n \subset U_n$ such that no $f \in V_i$ has a strong oscillation of length $i+1$. Each V_i is a weak Haar space.

(a) ⇒ (b) For $n = 0$ the statement is trivial.

$n-1 \Rightarrow n$. Suppose there exist an $h \in U_n \backslash U_{n-1}$ and $(t_0, ..., t_{n+1}) \in \Delta_{n+2}(M)$ with $h(t_0) < h(t_1) > h(t_2) < h(t_3) > \cdots$. Let $W = U_n|_{\{t_0, ..., t_{n+1}\}}$. If W

has dimension $<n+1$, different $(u_0,..., u_{n+1}) \in \Delta_{n+2}(M)$ and $\hat{h} \in U_n \backslash U_{n-1}$ may be constructed with $\dim U_n|_{\{u_0,..., u_{n+1}\}} = n+1$ and $\hat{h}(u_0) < \hat{h}(u_1) > \hat{h}(u_2) < \cdots$. This construction is completely analogous to the proof of Lemma 4.1 in [10], part "(b) $\Rightarrow$ (a)," Case 2.

So without loss of generality we assume $\dim W = n+1$. For each $f \in W$, let s_f be the linear spline with knots $t_0,..., t_{n+1}$ interpolating f in these knots. Clearly $X := \{s_f \mid f \in W\}$ is an $(n+1)$-dimensional normed weak Haar space of continuous functions on an interval and contains an element with a strong oscillation of length $n+2$. This, however, contradicts Zwick's result [12] which states Lemma 1 for continuous functions defined on an interval.

LEMMA 2. *Let $f_0, f_1,..., f_n$ be a normed weak Markov system, $p_0 := -\infty$, $p_{n+2} := \infty$, and $f \in U_n$ with a strong oscillation of length $n+1$ in $(p_1,..., p_{n+1}) \in \Delta_{n+1}(M)$, and $x, y \in M \cap (p_v, p_{v+1})$ for some $v \in \{0, 1,..., n+1\}$ with $f(x) = f(y)$. Then $h(x) = h(y)$ holds for all $h \in U_n$.*

Proof. Otherwise for some $\varepsilon \in \mathbb{R}$, $f + \varepsilon h$ would have a strong oscillation of length $n+2$ or $n+3$ in $p_1,..., p_v, x, y, p_{v+1},..., p_{n+1}$ ($n+2$ if $x < y < p_1$ or $p_{n+1} < x < y$).

LEMMA 3. *Let $f_0, f_1,..., f_n \in F(M)$ be a nondegenerate normed weak Markov system. If f_1 is constant on a subset $S \subset M$, every $f \in U_n$ is constant on S.*

Proof. For $n = 0$ and $n = 1$ the statement is trivial. Assume it holds for $n-1 \geqq 0$. There are l, $r \in M$ with $S \subset [l, r]$ for otherwise U_1 would be degenerate. As U_{n-1} is nondegenerate there exists a $g \in U_{n-1}$ with a strong oscillation of length n in $(t_1,..., t_n) \in \Delta_n(M)$ with $t_n < l$ and $g(t_{n-1}) < g(t_n)$. By Lemma 1 g is increasing on $M \cap [t_n, \infty)$. Let $u \in S$ be fixed. Let $\varepsilon_0 > 0$ such that $\hat{g} := g + \varepsilon_0 f_1$ has a strong oscillation of length in $t_1,..., t_{n-1}, u$, and $\hat{g}(t_{n-1}) < \hat{g}(u)$. There is a $t_{n+1} \in M \cap (r, \infty)$ with $\hat{g}(u) < \hat{g}(t_{n+1})$, and $\hat{g}$ is constant on S by induction hypothesis. If there were $f \in U_n$ and points $x, y \in S$ with $f(x) \neq f(y)$ say $x < y$ and $f(x) > 0 > f(y)$ without loss of generality, $\hat{g} + \varepsilon f$ would have a strong oscillation of length $n+2$ in $t_1,..., t_{n-1}, x, y, t_{n+1}$ for small $\varepsilon > 0$ in contradiction to Lemma 1.

DEFINITION. For $h \in F(M)$ and $a \in M$, let the right-hand limit of h in a with respect to M be defined by $M - \lim_{x \to a+} h(x) = \alpha$ iff for every $\varepsilon > 0$, there is a $z \in M$ with $a < z$ and $|h(x) - \alpha| < \varepsilon$ for all $x \in (a, z] \cap M$. The left-hand limit is defined analogously.

DEFINITION. Let $f_0, f_1,..., f_n \in F(M)$ be a normed weak nondegenerate Markov system, and for $a \in M$ let

$$L_a = \{x \in M \cap (-\infty, a) \mid f_1(y) \neq f_1(x) \text{ for all } y \in (-\infty, x) \cap M\},$$
$$R_a = \{x \in M \cap (a, \infty) \mid f_1(y) \neq f_1(x) \text{ for all } y \in (x, \infty) \cap M\},$$
$$l_a = \sup L_a, \qquad r_a = \inf R_a.$$

Then the left and right relative derivatives $(D_- f)(a)$ and $(D_+ f)(a)$ of $f \in U_n$ in $a \in M$ with respect to f_1 are defined by

$$\begin{aligned}(D_- f)(a) &= \frac{f(l_a) - f(a)}{f_1(l_a) - f_1(a)} \qquad \text{if} \quad l_a \in L_a \text{ and } l_a < a \\ &= M - \lim_{x \to l_a -} \frac{f(x) - f(a)}{f_1(x) - f_1(a)} \qquad \text{otherwise}\end{aligned}$$

and

$$\begin{aligned}(D_+ f)(a) &= \frac{f(r_a) - f(a)}{f_1(r_a) - f_1(a)} \qquad \text{if} \quad r_a \in R_a \text{ and } a < r_a \\ &= M - \lim_{x \to r_a +} \frac{f(x) - f(a)}{f_1(x) - f_1(a)} \qquad \text{otherwise,}\end{aligned}$$

if the corresponding expressions exist.

THEOREM 4. *Let $f_0, f_1, ..., f_n \in F(M)$ be a nondegenerate normed weak Markov system. Then*

(a) *for all $a \in M$ and $f \in U_n$ there exist $(D_+ f)(a)$ and $(D_- f)(a)$;*

(b) *$D_+ f_1, ..., D_+ f_n$ and $D_- f_1, ..., D_- f_n$ form nondegenerate normed weak Markov systems;*

(c) *for $P = \{x \in M \mid (D_+ f)(x) \neq (D_- f)(x)$ for some $f \in U_n\}$ there exist strictly increasing mappings i, s with $s \in F(M)$ and $i: P \to \mathbb{R} \backslash s(M)$ such that a nondegenerate normed weak Markov system $\phi_1, ..., \phi_n \in F(s(M) \cup i(P))$ is defined by*

$$\begin{aligned}\phi_v(x) &= (D_+ f_v)(s^{-1}(x)) \qquad \text{for} \quad x \in s(M) \\ &= (D_- f_v)(i^{-1}(x)) \qquad \text{for} \quad x \in i(P),\ v = 1, ..., n.\end{aligned}$$

Proof. For $n = 1$, the statements are trivial.

$n - 1 \Rightarrow n \geqslant 2$. (a) Obviously we may restrict our attention to $D_+ f$. For $a < r_a \in R_a$, the statement is trivial. For $a < r_a \notin R_a$, the statement follows from $(D_+ f)(a) = (D_+ f)(r_a)$. Now let $a = r_a$. Without loss of generality let f_1 be increasing, $f(a) = f_1(a) = 0$ and f be nonnegative on $M \cap (a, c)$ for a suitably chosen $c \in M \cap (a, \infty)$.

We first suppose

$$M-\lim_{x\to a+}\sup\frac{f(x)}{f_1(x)}=\infty.$$

By induction hypothesis for every $g\in U_{n-1}$ there exists

$$M-\lim_{x\to a+}\frac{g(x)-g(a)}{f_1(x)}.$$

For all $g\in U_{n-1}\setminus\{0\}$ with $g(a)=0$ and $x\in M\cap(a,\infty)$ with $g(x)\neq 0$ we have

$$\frac{f(x)}{f_1(x)}=\frac{f(x)}{g(x)}\cdot\frac{g(x)}{f_1(x)}.$$

As U_{n-1} is a nondegenerate weak Haar space there exist a $g\in U_{n-1}$ and $(t_1,..., t_n)\in\Delta_n(M)$ with $t_n=a$ and

$$\begin{aligned} g(t_i)&=(-1)^{n-i} && \text{for} \quad i=1,..., n-1\\ &=0 && \text{for} \quad i=n. \end{aligned}$$

By Lemma 1 g is increasing on $M\cap[a,\infty)$. By Lemma 2 there exists a $d\in M\cap(a,\infty)$ such that g is strictly increasing on $M\cap[a,d]$. By induction hypothesis we have

$$M-\lim_{x\to a+}\frac{g(x)}{f_1(x)}\geq 0,$$

thus

$$M-\lim_{x\to a+}\sup\frac{f(x)}{g(x)}=\infty.$$

So for all $\varepsilon>0$, we have

$$M-\lim_{x\to a+}\inf\frac{(g-\varepsilon f)(x)}{g(x)}=-\infty.$$

Thus, there is a $t_{n+1}\in(a,a+d)\cap M$ with $(g-\varepsilon f)(t_{n+1})<0<(g-\varepsilon f)(d)$. So for sufficiently small $\varepsilon>0$, $g-\varepsilon f$ has a strong oscillation of length $n+2$ in $t_1,..., t_{n+1}, d$ in contradiction to Lemma 1. Now suppose

$$\beta:=M-\liminf_{x\to a+}\frac{f(x)}{f_1(x)}<M-\limsup_{x\to a+}\frac{f(x)}{f_1(x)}=:\gamma.$$

Then

$$f - \frac{\beta + \gamma}{2} f_2$$

has a strong alternation of arbitrary length, a contradiction.

(b) This part of the proof is completely analogous to the proof of Theorem 11.3(b) in [10].

(c) We start with a lemma the proof of which is omitted since it is completely analogous to the proof of Lemma 11.1 in [10]:

LEMMA 4. *Let $f_0, f_1, ..., f_n \in F(M)$ be a nondegenerate normed weak Markov system. Then for $a, b \in M$, every $f \in U_n$ is bounded on $M \cap [a, b]$.*

For every $f \in U_n$, $D_+ f$ and $D_- f$ are piecewise monotone and bounded on every subset $M \cap [a, b]$, $a, b \in M$. So they are discontinuous in at most countably many points, and the set

$$P_f := \{x \in M \mid (D_+ f)(x) \neq (D_- f)(x)\}$$

is countable for every $f \in U_n$. From $P = P_{f_2} \cup P_{f_3} \cup \cdots \cup P_{f_n}$ follows that P is countable, say $P = \{p_1, p_2, ...\}$.

Let $d \in M$ be arbitrarily fixed, and $s \in F(M)$ be defined by

$$\begin{aligned} s(x) &= x + \sum_{\substack{v=1 \\ p_v \in (d,x]}}^{\infty} 2^{-v} \qquad \text{for} \quad x > d \\ &= x - \sum_{\substack{v=1 \\ p_v \in (x,d]}}^{\infty} 2^{-v} \qquad \text{for} \quad x \leqq d. \end{aligned}$$

Thus $s(M)$ has a "gap" of length 2^{-v} left of each $s(p_v)$. Let $i: P \to \mathbb{R}$ be defined by

$$i(x) = \tfrac{1}{2}(s(x) + \sup\{(-\infty, s(x)) \cap s(M)\}) \qquad \text{for} \quad x \in P.$$

So i inserts one point in the middle of each "gap" left of an $s(p_v)$. So $\phi_1, ..., \phi_n$ are well defined. Suppose there exist a $\phi \in \text{span} \{\phi_1, ..., \phi_n\}$, corresponding to an $f \in U_n$, and $(t_1, ..., t_{n+1}) \in \Delta_{n+1}(s(M) \cup i(P))$ with

$$(-1)^j \frac{f(s^{-1}(u_j)) - f(s^{-1}(t_j))}{f_1(s^{-1}(u_j)) - f_1(s^{-1}(t_j))} > 0 \qquad \text{if} \quad t_j \in s(M)$$

$$(-1)^j \frac{f(s^{-1}(u_j)) - f(i^{-1}(t_j))}{f_1(s^{-1}(uj)) - f(i^{-1}(t_j))} > 0 \qquad \text{if} \quad t_j \in i(P) \text{ for } j = 1, ..., n+1.$$

As f_1 is monotone, the set of points in M corresponding to $\{t_1, u_1, ..., t_{n+1}, u_{n+1}\}$ contains at least $n+2$ points forming a strong oscillation of f, a contradiction to Lemma 1.

For the proof of Theorem 3 we need the following result, which may be of some independent interest:

LEMMA 5. *Under the hypotheses of Lemma* 4, *every* $f \in U_n$ *is absolutely continuous with respect to* f_1 *on* $M \cap [a, b]$.

Proof. By Lemmas 1 and 4 every $f \in U_n$ may be continuously extended to $\bar{M} \cap (\inf M, \sup M)$. Now let $f \in U_n$ and $E := \bar{M} \cap [a, b]$. Because of Lemma 3 it is sufficient to show that

$$q(x, y) := \frac{f(y) - f(x)}{f_1(y) - f_1(x)}$$

is bounded for $x, y \in E$ with $f_1(x) \neq f_1(y)$. For $n \leqslant 1$, the statement is obvious.

$n-1 \Rightarrow n$. Suppose there exist $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ in E with $x_k < y_k$ and $q(x_k, y_k) \to \infty$ for $k \to \infty$. As f is bounded (Lemma 4) we have $f_1(y_k) - f_1(x_k) \to 0 (k \to \infty)$.

Without loss of generality let $x = \lim_{k \to \infty} x_k$ and $y = \lim_{k \to \infty} y_k$. As f_1 is monotone, say increasing, we have $f_1(x) = f_1(y)$, so $f_1|_{[x,y] \cap M}$ is constant. By Lemma 3, the set $[x, y]$ may be shrunk to one point, i.e., $x = y$. As U_{n-1} is nondegenerate, there exist $g \in U_{n-1}$ and $t_1, ..., t_n \in M$ with $t_1 < \cdots < t_n < x$ and $g(t_i) = (-1)^{n-i}$ for $i = 1, ..., n$. Thus g is increasing on $E \cap (t_n, \infty)$. Let $\varepsilon_0 > 0$ be such that $g := \hat{g} + \varepsilon_0 f_1 \in U_{n-1}$ has a strong oscillation in $t_1, ..., t_{n-1}, x$, and $\hat{g}(t_{n-1}) < \hat{g}(x)$. There is a $t_{n+1} \in M \cap (x, \infty)$ with $\hat{g}(x) < \hat{g}(t_{n+1})$. Let α be such that $\hat{f} := \hat{g} - \alpha f$ has a strong oscillation in $t_1, ..., t_{n-1}$ and $\hat{f}(t_{n-1}) < \hat{f}(x) < \hat{f}(t_{n+1})$.

For k sufficiently large, we have $t_{n-1} < x_k < y_k < t_{n+1}$, $\hat{f}(t_{n-1}) < \min\{\hat{f}(x_k), \hat{f}(y_k)\} \leqslant \max\{\hat{f}(x_k), \hat{f}(y_k)\} < \hat{f}(t_{n+1})$, and by induction hypothesis

$$\frac{\hat{f}(y_k) - \hat{f}(x_k)}{f_1(y_k) - f_1(x_k)} = \frac{\hat{g}(y_k) - \hat{g}(x_k)}{f_1(y_k) - f_1(x_k)} - \alpha q(x_k, y_k) < 0,$$

so $\hat{f}(x_k) > \hat{f}(y_k)$. But then $\hat{f}$ has a strong oscillation of length $n+2$ in $t_1, ..., t_{n-1}, x_k, y_k, t_{n+1}$.

Proof of Theorem 3. For $n = 0$ the statement is trivial.

$n-1 \Rightarrow n$. Let $\phi_1, ..., \phi_n$ be defined as in Theorem 4 and $N := s(M) \cup i(P)$.

Without loss of generality we assume that $f_1, \ldots, f_n$ are such that the induction hypothesis gives an integral representation (*) for $\phi_1, \ldots, \phi_n$ directly, i.e., without an additional linear transformation. So there exists a strictly increasing $h \in F(N)$ such that, with $J := (\inf h(N), \sup h(N))$, there exist increasing $w_2, \ldots, w_n \in C(J)$ and $c \in J$ such that $\phi_1 \equiv 1$ and

$$\phi_j(x) = \int_c^{h(x)} \int_c^{t_2} \cdots \int_c^{t_{j-1}} dw_j(t_j) \cdots dw_2(t_2) \qquad \text{for} \quad x \in N,\ j = 2, \ldots, n.$$

Moreover, we make the induction hypothesis that h is "gap-preserving," i.e., the gaps in N are transformed into corresponding gaps in $h(N)$ of equal length. Let $\tau_1 \equiv 1$ and

$$\tau_j(p) = \int_c^{p} \int_c^{t_2} \cdots \int_c^{t_{j-1}} dw_j(t_j) \cdots dw_2(t_2) \qquad \text{for} \quad p \in J,\ j = 2, \ldots, n,$$

so $\tau_j \in C(J)$ and $\tau_j \circ h = \phi_j$ for $j = 2, \ldots, n$. For $j = 0, 1, \ldots, n$ let $v_j \in F(J)$ be defined by

$$\begin{aligned} v_j(x) &= f_j(s^{-1} \circ h^{-1}(x)) && \text{for} \quad x \in h(s(M)) \\ &= f_j(i^{-1} \circ h^{-1}(x)) && \text{for} \quad x \in h(i(P)), \end{aligned}$$

v_j continuous on $J \cap \overline{h(N)}$, and on every subinterval Q of $J \backslash \overline{h(N)}$ let v_j be constantly equal to its value in the left endpoint of Q. Clearly $v_0, v_1, \ldots, v_n$ is a nondegenerate normed weak Markov system on J.

From Lemma 5 follows that each $v \in V := \operatorname{span}\{v_0, v_1, \ldots, v_n\}$ has the same continuity properties as v_1. Let $v_1 = v_c + j$ be the decomposition of v_1 where $v_c \in C(J)$ and j is a saltus function on J with $j(c) = 0$. Now let $k \in F(J)$ be defined by

$$k(x) = x + j(x) \qquad \text{for} \quad x \in J,$$

and $K := (\inf k(J), \sup k(J))$.

For every $g \in F(J)$ which is piecewise monotone and bounded on every compact interval of J, and is left-or right-continuous wherever v_1 is left-or right-continuous, let $\bar{g}\colon K \to \mathbb{R}$ be defined by:

$$\bar{g}(x) = g(k^{-1}(x)) \qquad \text{for} \quad x \in k(J),$$

$\bar{g}$ linear on every subinterval of $K \backslash k(J)$, and $\bar{g} \in C(K)$. So $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_n$ exist and form a nondegenerate normed weak Markov system on K. Also, we note that for $z \in C(J)$, on every subinterval of $K \backslash k(J)$, $\bar{z}$ is constant, so $d\bar{z} = 0$. Recalling $\tau_i, w_i \in C(J)$ for $i = 2, \ldots, n$ and $c = k(c)$ by definition of k, we thus get

$$\bar{\tau}_i(t) = \int_c^{t} \int_c^{t_2} \cdots \int_c^{t_{i-1}} d\bar{w}_i(t_i) \cdots d\bar{w}_2(t_2) \qquad \text{for} \quad t \in K \text{ and } i = 2, \ldots, n.$$

Besides, one checks that $(D_+\bar{v}_j)(t)=\bar{\tau}_j(t)$ holds almost everywhere with respect to the measure induced by $\bar{v}_1$. So Lemma 5 yields

$$\bar{v}_i(q)-\bar{v}_i(p)=\int_p^q \bar{\tau}_i(t)\,d\bar{v}_1(t) \qquad \text{for} \quad p, q\in K \text{ and } i=1,..., n,$$

and therefore

$$\bar{v}_i(q)-\bar{v}_i(c)=\int_c^q\int_c^{t_1}\cdots\int_c^{t_{i-1}} d\bar{w}_i(t_i)\cdots d\bar{w}_2(t_2)\,d\bar{v}_1(t_1), \qquad i=1,..., n.$$

For $x\in M$ we have $f_i(x)=(\bar{v}_i\circ k\circ h\circ s)(x)$, $i=0, 1,..., n$, and setting $r=k\circ h\circ s$, we get

$$g_i(x):=f_i(x)-\bar{v}_i(c)=\int_c^{r(x)}\int_c^{t_1}\cdots\int_c^{t_{i-1}} d\bar{w}_i(t_i)\cdots d\bar{w}_2(t_2)\,d\bar{v}_1(t_1), \qquad i=1,..., n,$$

i.e., an integral representation of the form $(*)$.

Proof of Corollary 3. If A has property (**B**), the restrictions of $D_+f_1,..., D_+f_n$ and $D_-f_1,..., D_-f_n$ to A form normed Markov systems (the argument is completely analogous to the one used for Theorem 11.3(c) in [10]). Thus, in the proof of Theorem 3, the functions $\phi_1,..., \phi_n$ form a normed Markov system on $s(A)$, and by induction hypothesis each $w_i|_{(h\circ s)(A)}$ is strictly increasing, $i=2,..., n$. $v_0, v_1,..., v_n$ for a normed Markov system on $(h\circ s)(A)$, so v_1 is strictly monotone. The same holds for $\bar{v}_1$ on $r(M)=(k\circ h\circ s)(A)$.

Acknowledgment

I would like to thank the referee for many helpful suggestions, which certainly improved this paper.

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